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ABSTRACT

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THE LEARNING OF ALGEBRA: A TEACHING EXPERIMENT

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THE LEARNING OF ALGEBRA: A TEACHING EXPERIMENT[1]

This paper focuses on the identification of some conceptual factors underlying the initial learning of algebra. It describes a study which uncovered three predominant notions existing in a sample of 12- and 13-year-olds prior to formal instruction in algebra, and the ways in which a particular approach to the teaching of algebra affected both the existing concepts and the construction of new ones. Analysis of protocol data indicates not only that certain aspects of the teaching outline provided powerful tools for transforming some of the pre-algebraic concepts, but also that contradictory notions seemed to coexist in the learner's conceptual framework and that some of these proved to be quite resilient to instruction.

Some Previous Studies

The investigation of conceptual factors underlying student performance in algebra is not altogether new, though the terminology often varies from study to study. A few studies have investigated pupils' understanding of the various symbols used in algebra (e.g., Küchemann, 1981; Matz, 1979). Many secondary school children view the equal sign as an operator symbol rather than as a symbol for equivalence (Kieran, 1981b). Some pupils have the misconception that changing a letter in an equation implies changing its value (Wagner, 1981). Researchers have also begun to focus recently on particular aspects of the equation solving process in order to specify more precisely the nature of solving strategies and errors (e.g., Carry, Lewis, & Bernard, 1980). It has been found that errors are not random; often they result from overgeneralization of rules to inappropriate domains (Davis,

[1] An abbreviated version of this paper is available.

Jockusch, & McKnight, 1978). Students who use intuitive strategies to solve equations usually find their methods insufficient when dealing with equations which involve negative integers (Bell, O'Brien, & Shiu, 1980). Pupils using a combination of formal and informal strategies generally seem to be more successful than those who use only one of these strategies (Petitto, 1979). Other studies have dealt with the translation of word problems into algebraic equations. Rosnick and Clement (1980) have shown that even engineering majors in college, who can often solve certain types of problems by their own intuitive methods, are generally unable to write an algebraic equation which corresponds to the relationship inherent in the problem.

The series of studies by Clement and his collaborators is especially interesting for its identification of two contradictory thought processes which can coexist in many students, that is, the use of letters to represent labels vs. their use as numerical variables (Clement, 1982). These two incompatible conceptualizations compete for dominance in certain task situations -- especially the translation of the "student-professor" word problem. Davis (1980) has identified three other commonly-held notions which underlie the learning of school arithmetic and which can account for seeming contradictory mathematical behavior among elementary school pupils when dealing with the symbolism of arithmetic: "the primary-grade undifferentiated binary operation frame", "the primary-grade addition frame", and "the primary-grade symmetric subtraction frame". The coexistence of competing structures in the student learning high school algebra has, however, never been investigated.

Aims of the Study

This study attempts:

1. To identify some of the pertinent conceptual notions held by 12- and 13-year-olds, prior to formal instruction in algebra (by means of a pretest conducted in individual interviews).
2. To trace changes occurring in these conceptualizations due to instruction (by means of a teaching experiment dealing with the early concepts in algebra, carried out with six individual subjects over a 3-month period, wherein the interviewer is also the instructor) -- the instructional outline based on a rational task analysis of elementary algebra.
3. To identify the components of the learner's conceptual framework which seem to be most benefitted by the planned teaching outline and those which seem to be resilient to instruction.
4. To chart the progress of notions which seem to be generated during the teaching experiment.
5. To compare the conceptualizations present at the end of the teaching experiment with those which were present at the beginning (by means of a posttest immediately after the teaching experiment -- in the month of June).
6. To assess the long-term effects of the teaching experiment on the changes wrought in the learner's conceptual framework (by means of a second posttest 3 months later in the month of September -- at the beginning of the next school year without any intervening instruction during the summer months).

A Note on Terminology and Method of Analysis

A method of analyzing the changing of conceptual schemes[1] over time has been suggested by Easley (1974). In his modeling of the cognitive structures used by children in certain mathematical and scientific tasks, he has used a technique developed by Witz (1973) which represents the changes occurring over time as different mental schemes become activated. This method aims at describing and documenting mental structures which the child has and which are specific to the child, with the underlying assumption that almost every interaction in the entire protocol of an interview may be accounted for by a small number of structures. This same technique of analyzing and diagramming the interplay among different conceptual schemes was used by Clement (1982) in his study mentioned above.

This method of examining sequences of verbal data in terms of the dominance of one scheme over another served as a useful heuristic device for analyzing the protocols of this study. Another advantage of this approach is that it is ideally suited for investigating changes in schemes, particularly the changes that occur as a result of instruction. Much of the past research in algebra has not had an instructional component. Thus, we have acquired a fairly large body of information on what students have or have not learned in algebra; considerably less on how they learn it. The inclusion of a teaching component in a sequence of one-on-one situations allows the researcher (who also does the teaching) to focus on the learner's extension of old schemes and construction of new ones at the very moments when the

[1]Piaget defines schemes as general structures of actions and operations, a means of generalization; he distinguishes these from schemas which, he says, are merely simplified imagined representations of the result of some organizatory activity.

learning is taking place, and thus provides for very precise information on how a particular conceptual scheme can change with respect to a specific form of instruction.

It is important to note that the term "schemes", as used in this paper, is not to be confused with "frames", "scripts", or "schemas".

The Study

In December, 1980, 10 students from the Secondary 1 level (12- and 13-years-old) were individually interviewed in order to identify some of their pre-algebraic notions (Kieran, 1981a). These children had not had any formal training in algebra. Their mathematics course at the Secondary 1 level consisted primarily of a review of the arithmetic concepts and skills taught in elementary school. They would begin algebra in Secondary 2. On interviewing these students it was found that 4 of them had already acquired some very good intuitions on algebraic symbols and expressions and were able to solve two-step linear equations in one unknown by a double-undoing process. These four students were not included in the rest of the study. The remaining six (who were all of average ability) were involved in a teaching experiment which was carried out in April, May, June, 1981.

The first two sessions with each individual in April did not involve any teaching. Subjects were asked questions which served as a pretest of their algebraic knowledge.[1] These same questions were repeated in two post-tests -- one at the end of June and another at the beginning of September. The numbers used in the equations were changed for the posttests. Sessions 3 to 9 consisted of the individual teaching-learning interviews, generally one per week. The teaching interviews included instruction in: i) extending the notion of arithmetic equality to include arithmetic identities; ii) introducing algebraic equations as arithmetic identities with a hidden number;

[1]See appendix for list of questions asked in prestest.

iii) looking at equations as the record of the steps in the "think of a number game"; iv) the order of operations convention; v) constructing meaning for the rule of performing the same operation on both sides of an equation; vi) a side-by-side treatment of "solving" an arithmetic identity with solving a corresponding algebraic equation; vii) the gathering of like terms; viii) solving equations without the aid of a set of parallel operations carried out on arithmetic identities. Nearly every session of the teaching experiment began and ended with the task of solving a set of 12 equations. The numbers and letters were changed for each occasion, but the forms of the 12 equations remained the same. All interview sessions were audiotaped.

Overview of the Three Conceptual Schemes

Before going into the details and evidence for the three main conceptual schemes which seemed to underlie the subjects' initial learning of algebra, they are outlined briefly first.

1. Quasi-equality scheme*: Based on a view of the equal sign which emphasized the operator notion rather than equivalence. It seemed to lead to two kinds of errors:

   a) Equations with an unknown on the left side were solved in terms of the first numeral on the right side, eg, $4 + \underline{x} - 2 + 5 = 11 + 3 - 5$ resulted in $4 + 4 - 2 + 5 = 11 + 3 - 5 = 9$.

   b) The Running Total Heuristic: It seemed to consist of the rule that, "It does not matter when you perform the operations, as long as they get totaled up sometime". The tacking-on of the last two operations after the unknown was evaluated (in the example above) is one manifestation of this

* In the abtreviated version of this paper, the quasi-equality scheme is split into two separate schemes: the Quasi-Equality scheme and the Running Total scheme.

rule. Another is that $3\underline{x} + 154 = 475$ is equivalent to $3\underline{x} = 475 + 154$ because "it does not matter if you add before the equal sign or after".

2. Redistribution scheme: Based on the notion that "taking something off one number and adding it on to another does not change anything", it seemed to account for the following error: $37 + \underline{b} = 168$ was transformed to $47 + \underline{b} = 158$. It also produced the following correct behavior, $3\underline{x} + 17 = 47$ transformed to $3\underline{x} + 5 - 5 + 17 = 47$.

3. Solving scheme: In general, this consisted initially in 4 sub-schemes (though differences were observed across subjects):

a) Number facts sub-scheme. For one-step equations with small coefficients, e.g., $3 + \underline{x} = 10$, use number facts.

b) Undoing sub-scheme. For one-step equations with large coefficients, e.g., $27\underline{x} = 540$, undo, i.e., $540 \div 27$.

c) Undoing-Substitution sub-scheme. For two-step equations, e.g., $3\underline{x} + 150 = 579$, undo first, then substitute, i.e., $579 - 150$, then "3 times what gives 429". (Some subjects substituted immediately in two-step equations.)

d) Substitution sub-scheme. For multistep equations with one or more unknowns, use trial-and-error substitution.

The next three sections provide evidence for the existence of these three schemes (using the protocols from one typical subject) and indicate how the teaching sequences aimed at constructing meaning for the concept of equation and thus allowing the subjects to alter their misconceptions. The third section which follows also indicates the generation of a fourth scheme -- the "gathering unlike terms" scheme (e.g., $7\underline{x} + 5$ can be replaced by $12\underline{x}$).

The Quasi-Equality Scheme

Initial State as Seen in the Pretest

The subject's initial quasi-equality scheme was based on the notion that the equal sign was an indicator to perform certain operations rather than a symbol for equivalence. Previous experience with equality statements in elementary school had usually consisted in evaluating the placeholder by either transposing the accompanying numerical term to the other side and changing its sign or substituting from known number facts. Furthermore, one side of these equality sentences most often contained a single numeral. Questions on the pretest which were slightly different from the very familiar one-step equations encountered in elementary school brought out aspects of the quasi-equality scheme which were rather unexpected, but which seemed to be due to the prominence of the subject's operator notion of the equal sign.

The pretest provided evidence of the notion that the value of the left side of an equation was equal to the first numeral on the right side.[1] After successfully solving equations such as $4\underline{c} + 3 = 11$ and $\underline{y} + 596 = 1282$, the subject was asked what he thought of $4 + \underline{x} - 2 + 5 = 11 + 3 - 5$:

1:179:INT*: What do you think of when you see that? ($4 + \underline{x} - 2 + 5 = 11 + 3 - 5$)

1:180:SUB: It could be, uh, . . . 4 plus that number would be, uh, ...(4)... that one's gotta equal four times, pick a number. Four plus ...

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*The first numeral refers to the session number; the second, the line in the protocol (for that session). Pauses of 1 or 2 seconds are indicated by 3 dots; longer ones, by the number of seconds (in brackets).

[1]The protocols yielded much evidence of the existence of the various schemes; however, only a few excerpts are presented in this paper.

5, 9, minus 2, 8, plus 5, that would be 13. So, 4 plus 7 would be 11, minus 2, would be 9, plus 5 would be 14.

1:181:INT: What are you heading for? What number are you aiming for?

1:182:SUB: For 11. ... Uh, 4 plus ... would be 6, 10 minus 2, no. 4 plus 3, minus 2, equals 7, minus 2 is 5, no. So, it's gotta be 4 plus 8, 12, minus 2, is 14, no, minus 2 is 7. Well, I think

1:183:INT: Let's start over. What are you trying for x?

1:184:SUB: 6. 4 plus 6 is 10, minus 2 is 8, plus 5, no. It's 4 plus 5 is 9, minus 2, is 7, plus 5, 12. So, just try, it's gotta be 4. (laugh) I went the wrong way. 4 plus 4, 8, minus 2 is 6, plus 5, 11, plus 3 is 14, minus 5, um, 9. 9. So, that would be 9.

1:185:INT: So, what is 9?

1:186:SUB: x. No, no, no. That 9 is all the thing, but x is 4.

1:187:INT: OK. You said x is 4.

1:188:SUB: Right.

1:189:INT: because 4 and 4 is 8, take away 2 is 6, plus 5 is 11.

1:190:SUB: 11.

1:191:INT: So, when you make x, 4, all of this comes out to be 11. Is that right?

1:192:SUB: Yes.

1:193:INT: And over here, you have 11 plus 3, is 14, take away 5 is 9.

1:194:SUB: Yes, that's it.

1:195:INT: Well, I'm curious to know why this side comes out to be 11, and this side comes out to be 11 and 3, 14, take away 5, is 9. So, this side comes out to be 9; this side comes out to be

1:196:SUB: 11.

1:197:INT: 11.

1:198:SUB: Yeh, and they say equal. But I don't think they mean like this equals this.

1:199:INT: Oh, you think it means this equals

1:200:SUB: 11.

1:201:INT: just the first one, eh.

The excerpt above also indicates one of the characteristics of the Running Total Heuristic: after solving for the unknown in terms of the first numeral on the right side, proceed to carry out the rest of the not-yet-performed operations of the right side. This component of the quasi-equality scheme showed up again later:

1:300:INT: What's the first thing you notice when you see this ($96 + 35c - 58 = 178 \div 2$)?

1:301:SUB: Well, the first thing I notice is ... they must have equalled, they could have been equal to this. These three would have ... probably equalled to this. And then that certain number that you get from all of this, you would have to divide by 2.

1:302:INT: So all of this would have to come out to 178?

1:303:SUB: Well, it could be. You can try 96 minus 58 plus, that that answer you would have to add it on to 35 times something and see if it goes to 178. And then you do divided by 2. That could be it.

It is clear that when the subject is asked to solve or to interpret equations which have other than the "answer" on the right side, he is not using the idea that both sides of the equal sign should have the same value. The numerals and operations on the right side after the first member have no bearing on the value assigned to the unknown. Thus, the subject ends up with an incorrect solution, but believes it to be correct because he has performed all the operations in the sentence. It does not seem to matter very much when the operations on the right side are performed. This notion was also indicated in another type of task which did not demand the actual solution of equations, but rather a comparison of equivalent or non-equivalent equations:

2:1:INT: If you see this, $\boxed{297 + a = 734}$, and I show you another one, $297 - 297 + a = 734 + 297$, can you tell just by looking at them if this a is going to have the same value as that a?

2:2:SUB: Um ... (6) ... Well, this is zero, plus a equals 734 plus 297. Well, it could be the same thing. It could be, cause here's zero, so if you take it off, it will be like plus a equals, this is added up after the equal sign, here it's still before.

2:3:INT: So, do you think the a will be the same here?

2:4:SUB: Yes.

2:5:INT: as it is there? It will be the same number?

2:6:SUB: Yes.

Effects of Teaching Sessions on Quasi-Equality Scheme

Brief summary of the teaching. The introduction to the concept of an algebraic equation was based on the use of arithmetic identities (Herscovics & Kieran, 1980; Kieran 1981b) -- arithmetic equalities with multiple operations on both sides. Using this approach, the concept of equality and the corresponding use of the equal sign were expanded within the domain of arithmetic equalities. Subjects first constructed multi-operation arithmetic identities (e.g., 3 + 5 + 7 - 2 = 11 + 2). These were then transformed into algebraic equations by "hiding one of the numbers". This particular teaching method makes the idea of maintaining the balance on both sides of the equal sign very explicit. Thus, an equivalence notion of the equal sign can begin to take priority over an operator notion.

In constructing his arithmetic identities, the subject was evaluating each side to verify the presence of the same value. Later, when solving equations, he no longer evaluated the unknown in terms of only the first member of the right side:

3:253:INT: How would you do this one? (3 + x - 1 + 7 = 19 - 5 + 2)

3:254:SUB: First you do the first side, which is 17.

3:255:INT: Where is 17?

3:256:SUB: Here. This side here.

3:257:INT: How did you get 17?

3:258:SUB: 19 minus 5 equals 15, plus 2 is 17. So, this answer's, this side's gotta come up to 17.

This excerpt would seem to indicate also the disappearance of part of the Running Total Heuristic -- having already used up the "dangling" operations on the right hand side, the subject no longer tacked them on to the end of his sequence of calculations on the left hand side.

Short-term effects of teaching sessions on the quasi-equality scheme. The short-term effects of the teaching sessions were measured by a posttest given in the month of June at the end of the teaching experiment. It was another interview situation, with the same questions as were asked in the pretest three months earlier (only the coefficients of the equations to be solved were changed).

In solving equations with multiple operations on both sides, the subject never attempted during the posttest to evaluate the left side in terms of only the first numeral on the right side. Thus, the "dangling operations" aspect of the Running Total Heuristic seemed altered. However, another part of the RTH seemed relatively untouched by the teaching experiment:

8:256:INT: If you had $15x - 264 = 376$ and suppose you started off to solve it this way, $15x = 376 - 264$, would you be alright?

8:257:SUB: Uhm, ... (5) ... no.

8:258:INT: Why not?

8:259:SUB: 'Cause you're not doing the same thing to both sides (N.B.: Part of the teaching experiment had focused on the use of the rule of "doing the same operation to both sides".). Well, I mean. Here before it's 15 minus. Yeah, it would be the same thing. 'Cause here (in the original) you subtracted before getting the answer. Here (in the second equation) you're subtracting at the answer. It's just you're moving it back.

When asked to interpret, rather than solve, two equations, the subject still seemed to allow for performing an operation either just before the equal sign or just after. If we compare this response with that given to the same question asked in the pretest, we see the same kind of justification being offered on the two occasions:

2:88:INT: If you have to find out what x is in this, $\boxed{15x - 264 = 376}$, and suppose you did this, $15x = 376 - 264$, would you be alright, or would you be doing something wrong?

2:89:SUB: I think it would be alright. You're just changing the question around.

2:90:INT: Do you think x will have the same value, even though the question is changed around?

2:91:SUB: Yeah. Yes.

2:92:INT: What if someone did this, $15x = 376 + 264$, would the x still be the same, or are they making a mistake in doing that?

2:93:SUB: I think they are, cause

2:94:INT: They are what?

2:95:SUB: Doing a mistake, cause here they took $15x$ equals 376, but here they're taking off from $15x$, 264, but here they're adding on 264 to 376. They should be taking off, from this or from this.

In contrast to this response, another similar question on the posttest evoked a different scheme:

8:190:INT: If we have $\boxed{427m + 130 = 852}$, will this one, $427m = 852 + 130$, have the same solution?

8:191:SUB: No. You're supposed to subtract 130 from 852.

The issue of the existence of these contradictory schemes will be discussed in the last section of the paper.

Long-term effects of teaching sessions on the quasi-equality scheme. The September posttest was conducted immediately at the beginning of the

next school year. Subjects had not had any algebra lessons during the summer. The subject whose protocols are being used in this paper continued to solve equations with multiple numerical operations on the right hand side by evaluating the right hand side prior to solving for the unknown. His view of equality still included the notion that both sides had to have the same numerical value prior to and after some transformation.

The sub-scheme that allowed for the performance of an operation either before the equal sign or just after (the Running Total Heuristic) did not become dominant at all during the September posttest:

9:235:INT: If you have $\boxed{15\underline{x} - 264 = 376}$, would it be alright to start off by doing this, $15\underline{x} = 376 - 264$?

9:236:SUB: No, you have to do the same thing to both sides.

9:237:INT: And what did they do here?

9:238:SUB: They changed it around, like they put it on the other place. They should have done it plus 264 and here plus 264.

In fact, he seemed to be quite explicit in denying the RTH when interpreting the equivalence of two one-step equations:

9:146:INT: If we have this as our model equation, $\boxed{560 + \underline{b} = 683}$, and this other one, $560 = 683 + \underline{b}$. Can you tell me if the $\underline{b}$'s are the same in both of them?

9:147:SUB: No

9:148:INT: Why not?

9:149:SUB: Because here the $\underline{b}$ is added to the answer, and there it's added to the addend.

Thus, we can conclude that, three months after the teaching experiment, the subject's quasi-equality scheme has altered in some very clear ways: a) multi-step equations are no longer solved in terms of the first numeral on the right hand side; b) consequently, there are no "dangling

operations" left undone on the right hand side, waiting to be tacked on afterward. It is still not certain whether the part of the Running Total Heuristic which states that, "when comparing equations, it does not matter whether or not a particular operation is carried out before the equal sign or after", is still present, but lying dormant, or not.

The Redistribution Scheme

Initial State as Seen in the Pretest

The identification of this scheme is seen as one of the main original contributions of this paper. The subject, whose protocols are being used in this paper, provided clear evidence of the existence of this scheme. The subject's initial redistribution scheme was characterized by the underlying notion that whatever was subtracted from one number was to be added to another and vice versa. The location of the two transformed numbers seemed irrelevant. Thus, we see in his protocols all of the following transformations:

(i) $\underline{a} + \underline{x} = \underline{b} \Rightarrow \underline{a} + \underline{x} + \underline{c} = \underline{b} - \underline{c}$ (where $\underline{x}$ represents the literal term, and $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$ represent numerical terms)

(ii) $\underline{a} + \underline{x} + \underline{b} = \underline{c} \Rightarrow (\underline{a} - \underline{d}) + \underline{x} + (\underline{b} + \underline{d}) = \underline{c}$

(iii) $\underline{ax} + \underline{b} = \underline{c} \Rightarrow (\underline{a} - \underline{d})\underline{x} + (\underline{b} + \underline{d}) = \underline{c}$

This scheme also included the concept that adding a quantity to and then subtracting it from a given number (the adding-zero sub-scheme) changes nothing, i.e.,

(iv) $\underline{a} + \underline{x} = \underline{b} \Rightarrow \underline{a} + \underline{c} - \underline{c} + \underline{x} = \underline{b}$.

One can argue that the basis of most of this scheme is an overgeneralization of either (a) an undoing scheme learned in early elementary grades, e.g., $\underline{n} + 5 = 8 \Rightarrow n = 8 - 5$, wherein the addition of 5 on the left side of the first equation leads to the subtraction of 5 on the right side of

the second equation; or (b) the adding zero sub-scheme. However, it can also be argued that its roots may lie deeper than this. It could be based on a redistribution scheme of everyday life. We can envision the following scenario: Three children dipping into a bag of candy and pulling out 5, 3, and 4 candies respectively. One child (perhaps the one who pulled out 3) suggests that the child with 5 candies give away 1 of his to the child with only three. Then the candies become more evenly distributed. In one sense nothing has changed; the total number of candies has remained the same. This sharing scheme suggests a non-symbolic origin of the redistribution scheme, or at least one in which the equal sign is completely irrelevant. Though we can symbolize the sharing transformation as $\underline{a} + \underline{b} + \underline{c} = \underline{d} \Leftrightarrow (\underline{a} - \underline{e}) + (\underline{b} + \underline{e}) + \underline{c} = \underline{d}$, the protocols would seem to indicate that when the focus is on this type of transformation, i.e., redistribution, the equal sign is not being attended to.

The first occasion on which the subject's redistribution scheme surfaced was in response to the following question:

1:223:INT: When you see that (a card with $3\underline{a} + 156 = 372$ written on it), can you think of something to do to that equation that will make $\underline{a}$ change?

The subject answered that "he didn't know how to make the $\underline{a}$ change", but he could do something "where the $\underline{a}$ would stay the same"

1:226:SUB: You can do, uh, like ... (7) ... like take off some numbers like this 6, put it to zero and, hum, the 6 that's here you can put it, you can add it to here. Well, it'll still come to the same number when you times this by this, And you divide by this, it'll still come to this one. So, if you take the 6 here, $9\underline{a}$ plus 150 (he wrote: $9\underline{a} + 150 = 372$)

He was suggesting that $9\underline{a} + 150 = 372$ was equivalent (our term, not his) to $3\underline{a} + 156 = 372$. This was the "adding on to one number and subtracting

from another number but all on the same side" aspect of his redistribution scheme. There was no discrimination being made as to the possible constraints posed by the multiplication.

A little later in the pretest, the subject was shown the equation, $37 + \underline{b} = 168$, and was asked if he could make up another equation that would have the same solution as the given one:

1:316:INT: Could you make up another equation that will have the same solution as this, $37 + \underline{b} = 168$? Do you know what I mean by solution?

1:317:SUB: Yeah.

1:318:INT: The same value of $\underline{b}$. Could you make up a different equation that will have a $\underline{b}$ in it and the $\underline{b}$ will stand for the same number.

1:319:SUB: Yes. Well, we can, uh, do 168 and take some uh ... some of the 168, take it off and put it here.

1:320:INT: Do you want to show me?

1:321:SUB: Well, you could do 158 and uh that would be, then you would do 47, 47 plus $\underline{b}$ equals 158. (Wrote: $47 + \underline{b} = 158$)

1:322:INT: And that will have the same solution as this one? How do you know?

1:323:SUB: Well, cause, uh, you just took, transferred the 10 from here to here.

This excerpt indicates the "add on to one side and subtract from the other side" component of the subject's redistribution scheme. Both this task and the previous one demanded that the subject generate equivalent equations; later tasks asked him to compare two given equations and decide by looking at them whether or not they were equivalent. Production tasks always demand more of the subject than interpretation tasks. Thus, the uncovering of the redistribution scheme on these production tasks gives strong evidence for its existence. In contrast, the interpretation

task -- that of comparing two equations and deciding on their equivalence -- often invoked other schemes, such as the undoing scheme, which will be seen later.

A couple of examples from the set of equivalent-equations-interpretation task indicate other aspects of the subject's redistribution scheme. He was asked whether he thought the a in the given equation, 297 + a = 734, had the same value as the a in 297 + a - 734 = 734 - 734:

2:8:SUB: ... No, I don't think it will be the same thing.

2:9:INT: Why not?

2:10:SUB: Cause 297 plus a equals here 734, that's the answer. Here it's minus 734 equals 734. There should be a plus sign to get it back, like here. 'Cause here you're taking it off. So, it would be plus 734 would equal 734 here. (He was suggesting: 297 + a - 734 + 734 = 734).

In this example the subject is stating that the second equation is not equivalent to the first because, "there should be a plus sign to get it (the 734) back" on the left hand side. In other words, it should read, 297 + a - 734 + 734 = 734; when 734 is subtracted, it must then be added back on again. The resulting value of 0 would then leave the rest of the equation in the same state as the given one. This "adding zero" scheme was often invoked, but especially so later on , during the September posttest.

In a task assessing subject's ability to pick out errors in the first step of an equation-solving process:

2:66:INT: Somebody started to solve this equation, 6 + 39m - 4 + 2 = 43. This was the way they started off, 6 + 35m + 2 = 43. Is it alright to do that or not?

2:67:SUB: Well, ... (4) ... I guess so. Cause, uh, from here took it off and put it here instead. There was a 4 missing here and he added 4 on here. So he took off 4 from here, and just transferred it here.

Again we see an example of the process of transferring a number from one term to another.

Effects of Teaching Sessions on Redistribution Scheme

Brief summary of the teaching. After having learned to construct arithmetic identities (in order to give explicit meaning to an equivalence notion of the equal sign) and to transform these into equations, the subject began to work with the process of "doing the same operation to both sides" (DSOTBS) of an arithmetic identity. This was then applied to the solving of an equation generated from the arithmetic identity. Arithmetic identities and equations were "solved" in parallel using the technique of DSOTBS , for example:

$$15 \times \boxed{3} + 12 = 57 \qquad\qquad 15\underline{a} + 12 = 57$$
$$15 \times \boxed{3} + 12 - 12 = 57 - 12 \qquad\qquad 15\underline{a} + 12 - 12 = 57 - 12$$
$$15 \times \boxed{3} = 45 \qquad\qquad 15\underline{a} = 45$$
$$\frac{15 \times \boxed{3}}{15} = \frac{45}{15} \qquad\qquad \frac{15\underline{a}}{15} = \frac{45}{15}$$
$$\boxed{3} = 3 \qquad\qquad \underline{a} = 3$$

This side-by-side method allowed the subject to constantly check the validity of whatever operation he was performing. Later, equations were solved without the crutch of the arithmetic identities.

During the teaching-learning sessions, the subject never used his redistribution scheme. Yet it seems quite clear that the use of the rule of DSOTBS took over parts of the redistribution scheme. This conclusion is based on a comparison of the subject's responses on the two posttests with his responses on the pretest.

Short-term effects of teaching sessions on redistribution scheme. The June posttest indicates the short-term effects of the teaching sessions on the subject's redistribution scheme:

8:116:INT: If you have this equation, $3\underline{a} + 156 = 372$, could you do something to this equation that will change the $\underline{a}$, that will make the $\underline{a}$ change in value from what it is now?

8:117:SUB: 3 plus 156$\underline{a}$ equals 372. $(3 + 156\underline{a} = 372)$

8:118:INT: Could you do something to this equation that won't change the $\underline{a}$?

8:119:SUB: Oh, 3$\underline{a}$ plus 156 plus 6 equals 372 plus 6. $(3\underline{a} + 156 + 6 = 372 + 6)$

8:120:INT: Would you consider that equation that you just gave me to be the same as this one, or different?

8:121:SUB: Well, same, same cause you're doing the same thing to both sides.

8:122:INT: What about this equation, 26$\underline{c}$ = 594. Could you do something to that equation that will change the solution? That will change the value of

8:123:SUB: 26 equals 594$\underline{c}$. $(26 = 594\underline{c})$

8:124:INT: Could you do something that won't change the solution?

8:125:SUB: 26$\underline{c}$ plus 5 equals 594 plus 5. $(26\underline{c} + 5 = 594 + 5)$

This excerpt indicates that certain erroneous parts of the subject's redistribution scheme seem to have changed, for example, the shifting of a factor of a product to another term, shifting a factor from one side of the equation to the other. (We can presume that the teaching session on the order of operations facilitated this change.). Furthermore, the subject no longer suggested adding a value to one side and subtracting it from the other as he had in the pretest ($37 + \underline{b} = 168 \Rightarrow 47 + \underline{b} = 158$). The notion of "doing the same operation to both sides" (DSOTBS) seemed to have replaced this component of his redistribution scheme, $37 + \underline{b} = 168 \Rightarrow 37 + \underline{b} + 6 = 168 + 6$.

In the task on interpreting equivalent equations, for example, $297 + \underline{a} = 734$: in the pretest, the subject had answered that $297 + \underline{a} - 734 = 734 - 734$ was not right because the "-734" on the left hand side had to be followed by "+734" in order "to get back the -734"; in the posttest, the DSOTBS scheme was being invoked quite often and was being used as the first criterion for equivalent equations. Thus, in the post-

test, the subject answered that 297 + a - 734 = 734 - 734 was right "cause you're doing the same thing to both sides" (8:195).

Long-term effects of teaching sessions on redistribution scheme. On one type of question in the September posttest there was a return to part of the redistribution scheme, a part which had not appeared in June because of the strength of the DSOTBS scheme.

9:55:INT: Suppose you have x + 469 = 1351 and suppose I add 10 to that (the left hand side). Will this x (in x + 479 = 1351) still work out to the same as this x?

9:56:SUB: After you have worked it out?

9:57:INT: When you work out this

9:58:SUB: No.

9:59:INT: Why?

9:60:SUB: Because here you have 10 more.

9:61:INT: Well, can I do anything else to this equation so that this x will work out to be the same as this x?

9:62:SUB: You minus 10 here (pointing to 1351).

9:63:INT: Subtract 10? You mean make this 41?

9:64:SUB: Yes. (Wrote: 1341 -- by crossing out 5 in 1351)

9:65:INT: So, x plus 479 equals 1341 (x + 479 = 1341), will the x work out to be the same thing as the x in this one, x + 469 = 1351, is that what you are saying?

9:66:SUB: ... Yes.

When the interviewer suggested the adding-on of 10 to one side, the subject suggested then subtracting the 10 on the other side. On the next task, the subject was asked to generate his own equation, one that had the same solution as 3a + 156 = 372:

9:72:SUB: Well, 3a + 156 + 2 - 2 = 372 + 2 - 2 .

9:73:INT: Alright, what about this one, 26c = 594. Could you give me an equation where the c will come out to be the same thing as the c here.

9:74:SUB: 26c plus 5, no , plus 15 minus 5

9:75:INT: Do you want to write it down?

9:76:SUB: OK. (Wrote: 26c + 15 - 5 - 10 = 594 + 15 - 5 - 10)

What is interesting about these last two "production" examples is the way in which the DSOTBS and adding-zero schemes sifted together during the summer-months' absence from school. The DSOTBS scheme is still quite strong, but it has incorporated the adding-zero part of the redistribution scheme (at least in the examples generated by the subject). This is shown also in the following excerpt:

9:132:INT: Could you make up an equation which will have the same solution as this one, 37 + b = 168? ...

9:133:SUB: Add on and subtract the same number to each side.

9:134:INT: Like what, give me an example.

9:135:SUB: 37 + 12 - 12 + b = 168 + 12 - 12, that way it's zero.

9:136:INT: You are adding zero on to both sides.
What about this one, 28y = 362.

9:137:SUB: You would ...

9:138:INT: Could you make up another equation that will have the same value of y?

9:139:SUB: ... Oh ya. You could do the same thing like before, 28 plus 0 equals 362 plus 0, something that equals zero on both sides.

9:140:INT: Does it have to equal zero on both sides?

9:141:SUB: Oh, ya, it could be the same numbers too.

9:142:INT: Like what?

9:143:SUB: Like 6 on both sides, you can add 6.

The preceding examples make one wonder just how much of the redistribution scheme is still evokable. When the interviewer suggested adding 10 to one side of an equation, the subject came back with subtracting 10 on the other side. Yet, when he constructed his own equivalent equations, he combined both the DSOTBS and adding-zero techniques. In the next example where the model equation and the equivalent (or non-equivalent) alternative were provided in written form, the subject indicated that "adding on one side and subtracting on the other side" was not permissible:

9:158:INT: If we have this as our model equation, $\boxed{560 + b = 683}$, and this other one, $b + 560 + 125 = 683 - 125$, can you tell me if the b's are the same in both of them?

9:159:SUB: No, because you are not doing the same thing to both sides. Here you are adding 125 and here you are subtracting 125.

Other examples in this task indicated equally strongly that the "adding on one side and subtracting from the other side" scheme was not being called up, nor was the "shifting of a factor from one term to another".

9:163:INT: Alright, if we start with this, $\boxed{427n + 130 = 852}$, is this one, $427 + 130n = 852$, going to have the same value of n?

9:164:SUB: No, because you can't change the n from the numbers, like the answer (852) will be the same, but the n won't be the same.

Also:

9:175:INT: If we start with that, $\boxed{297 + a = 734}$, will this one have the same value of a, $297 - 297 + a = 734 + 297$.

9:176:SUB: ... (8) ... No, because you're not doing the same to both sides.

9:177:INT: What are we doing here?

9:178:SUB: You are adding on one side the 297, on the other side you are subtracting.

9:179:INT: How about that one, $297 + a - 734 = 734 - 734$ (to be compared with $297 + a = 734$)?

9:180:SUB: Yes, because you are doing the same thing to both sides again, subtracting 734.

What can we conclude about the status of the redistribution scheme? When the subject was provided with two written equations (the model and its alternate form), he stated that "adding on one side and subtracting on the other" was not permissible. This would indicate that the first part of his redistribution scheme seems altered. However, when only one equation was given in written form and the interviewer asked about adding a number on the left side, the subject suggested subtracting that number from the right side. Furthermore, in the task which followed, a generation task also, the subject used the DSOTBS technique to construct an equation with the same solution. It would seem that both the redistribution scheme and the DSOTBS scheme exist, and simply that one or the other dominates at a given moment. This implies that the teaching of DSOTBS rule did not remove or destroy the "add to one side, subtract from the other" rule, but left it free to emerge again at any time. This implies further that the basis of the "add to one number, subtract from another" notion may be very deep, making it quite resilient to instruction. Thus, the last word has obviously not yet been said on the teaching approach that would facilitate the transformation of this seemingly deep-seated scheme.

The Solving Scheme

Initial State as Seen in the Pretest

In the April pretest, the subject indicated that his scheme for solving one-step equations consisted of undoing (undoing sub-scheme) for equations with large coefficients, that is, performing the inverse operation, $x + 15 = 37 \Rightarrow x = 37 - 15$, and using number facts for one-step equations with smaller coefficients. This was the type of equation he had had much

experience with in elementary school. During the pretest, he was also asked to solve equations with more than one step. He had had very little prior experience, it would seem, with these other types. Simple two-step equations, e.g., $2\underline{x} - 6 = 4$, were attempted initially by a trial-and-error substitution (substitution sub-scheme). In the last equation-solving problem of the pretest, he seemed to alter his solving scheme slightly, so that equations like $4\underline{c} + 3 = 11$ were solved partially by undoing, partially by substitution -- "11 minus 3 equals 8, and 4 times 2 equals 8" (undoing-substitution sub-scheme). Equations with more than two steps, such as, $4 + \underline{x} - 2 + 5 = 11 + 3 - 5$, were solved by substitution. As we have already seen, the subject substituted different values for $\underline{x}$ until the left hand side had a value of 11. Other multistep equations, such as, $3\underline{a} + 5 + 4\underline{a} = 19$ or $2\underline{c} + 5 = \underline{c} + 8$, were attempted by means of substitution.

Thus, the subject's solving scheme consisted of 4 sub-schemes:

i) Using number facts -- for one-step equations with small coefficients;

ii) Undoing scheme -- for one-step equations with large coefficients;

iii) Undoing-Substitution scheme -- for some two-step equations;

iv) Substitution scheme -- for equations with two or more steps.

Equation-solving tasks called up one of these solving sub-schemes, depending on the form of the equation. However, even tasks which did not demand the solving of equations explicitly, such as, the comparison of equivalent equations task, sometimes made use of the undoing solving scheme:

1:398:INT: Let's take this as a model, $\boxed{297 + \underline{a} = 734}$. I'm going to show you four equations and I want you to tell me whether the $\underline{a}$ in those equations will have the same value as the $\underline{a}$ in our model. How about $297 - 297 + \underline{a} = 734 + 297$, will the $\underline{a}$ have the same value?

1:399:SUB: ... (3) ... Nope. No.

1:400:INT: Why not?

1:401:SUB: 'Cause like this is zero plus a, like these two (the 734 + 297) are supposed to be like minus here, they're supposed to be 734 - 297.

However, when this same question was repeated at the beginning of the next session, the subject answered on the basis of his quasi-equality scheme:

2:1:INT: If you see this, $297 + a = 734$, and I show you another one, $297 - 297 + a = 734 + 297$, can you tell just by looking at them if this a is going to have the same value as that a?

2:2:SUB: Um ... (6) ... Well, this is zero, plus a equals 734 plus 297. Well, it could be the same thing. It could be, cause here's zero, so if you take it off, it will be like plus a equals, this is added up after the equal sign, here it's still before.

2:3:INT: So, do you think the a will be the same here?

2:4:SUB: Yes.

2:5:INT: as it is there? It will be the same number?

2:6:SUB: Yes.

The question as to why two different schemes were invoked for the identical task -- the undoing scheme at the end of the first session; the quasi-equality scheme at the beginning of the second session -- will be discussed at the end of the paper.

Effects of Teaching Sessions on Solving Scheme

Brief summary of the teaching. Since the teaching on equation solving took several sessions, this section will include many of the subject's reactions to different aspects of the teaching outline. The beginning and end of each session usually consisted of a set of equations which were to be solved. This "equation template" was supposed to register the effect of any of the teaching approaches on the processes used in solving equations. These templates all had the same form -- only the numerals were different on

each occasion.

Accordingly, the session involving the construction of arithmetic identities and their transformation into equations was followed by an equation template. In solving the fourth equation (following $7\underline{m} = 28$, $3\underline{b} - 5 = 16$, and $\underline{x} + 421 = 978$), $18\underline{y} - 132 = 372$, after the subject had added 372 and 132, it was greatly emphasized that 504 was the value of $18\underline{y}$ (in the manner of Whitman, 1975; Adi, 1978). This explicitness was judged necessary because on previous occasions where the subject had been asked, for example, "What is it that has a value of 504?", the only answer that he could come up with was "the sum of 372 and 132". Repetition of this question in the remaining two-step equations of the template led the subject to enlarge his undoing scheme to include two-step equations. However, multistep equations, such as, $8 + \underline{x} + 1 + 6 = 29 - 5 + 2$, were still being solved by substitution, even though the subject had condensed both left and right sides into a one-step equation.

Immediately after constructing meaning for the rule of doing the same operation to both sides of an arithmetic identity, the subject was asked to solve a set of equations. It was hypothesized that he would spontaneously alter his equation-solving processes using this new rule, though he had not yet received any specific instruction on how it might be helpful to do so. He did not attempt to apply the new rule to the equation template.

The instruction which followed consisted of a parallel treatment of arithmetic identity and equation, both being solved by the rule of doing the same operation to both sides. An example showing this method was given in the previous section (page 19). Thus, the subject was now being

shown how to solve equations by the rule of doing the same operation to both sides -- the same type of equation that he had already been solving by "undoing twice". The subject did remark that this method seemed quite time-consuming; so, the link between this method and his undoing method was shown explicitly.

The subject began some time later to work with equations involving two occurrences of the unknown, for example, $24\underline{a} + 7\underline{a} + 59 = 741$. After subtracting 59 from both sides, he was left with $24\underline{a} + 7\underline{a} = 682$. He then suggested dividing both sides by 24 and then by 7. Instruction in the addition and subtraction of like literal terms, using a concretization of the notion of distributivity, then followed. After solving for $31\underline{a} = 682$, the subject asked whether he should "split the 22 (the solution), in two" and substitute 11 rather than 22 in the original equation, since it contained two occurrences of the unknown rather than one. Since this equation was not worked out in parallel with an arithmetic identity, it is not known if the same question would have arisen in a side-by-side treatment.

The next equation, which was worked out in parallel with an arithmetic identity, was one involving an occurrence of the unknown on each side of the equation, $37\underline{x} + 55 = 25\underline{x} + 139$. After subtracting 55 from both sides, subject was left with $37\underline{x} = 25\underline{x} + 84$, with no idea of what he might try next. The rule of operating on both sides (with numerical values) did not seem to be transferring spontaneously to the same procedure with literal values. The subject had to be guided into subtracting $25\underline{x}$ from both sides. This ended the teaching-learning session that day, with no time left for an equation template. In the equation-solving template which began the next session a few days later, all two-step equations were solved by a

double undoing -- just as the subject had solved them prior to the previous instructional session.

The first equation with two occurrences of the unknown (of the same template), $3\underline{a} + 5 + \underline{a} = 21$, the subject tried to solve by the rule of doing the same operation to both sides (rather than by trial-and-error substitution, as he had done previously). Two interesting errors occurred:

$3\underline{a} + 5 + \underline{a} = 21$	line 1
$3\underline{a} + 5 + \underline{a} - 5 = 21 - 5$	line 2
$3\underline{a} + \underline{a} = 16$	line 3
$3\underline{a} + \underline{a} - 3 = 16 - 3$	line 4
$\underline{a} + \underline{a} = 13$	line 5
$1\underline{a} = 13$	line 6

In line 4, he did not gather the like literal terms; instead he subtracted 3 from both sides. He then simplified $3\underline{a} - 3$ to $\underline{a}$. There was no event in the previous session that had hinted at this. Though it is a well-known and well-documented error, the subject had never displayed this behavior before. Therefore, how was this simplifying process constructed? In the previous session, the subject had not adopted the notation that the instructor used when dividing both sides by the same quantity. Instead of using the format, $\frac{7\underline{a}}{7}$, subject wrote $7\underline{a} \div 7$ (perhaps because of some aversion to the fractional form for division). The $7\underline{a} \div 7$ was replaced on the next line by $\underline{a}$. The question arises as to whether this horizontal notation facilitates the adoption of the rule that allows $3\underline{a} - 3$ to be replaced by $\underline{a}$. (Further discussion of the spontaneous generation of this scheme will take place in the last section of the paper.). The erroneous (line 6) replacement of $\underline{a} + \underline{a}$ by $1\underline{a}$ resembled to a certain extent the reverse of the subject's suggestion the week before that a particular solution should be split in two because of the two occurrences of the unknown.

On the next equation that he was asked to solve, $3 \times \underline{y} + 2 = 4 \times \underline{y} + 1$, he reverted to his substitution scheme (presumably because of his lack of success using the rule of DSOTBS), likewise for the solving of the equation, $3\underline{d} + 3 = 4\underline{d}$. After further instruction, the subject altered his DSOTBS scheme (when applied to the solving of multistep equations) in such a way that the final step of the solving process involved the undoing scheme. This seemed to indicate a partial integration of the DSOTBS scheme and the undoing scheme.

Short-term effects of teaching sessions on solving scheme. The June posttest began with the usual template of equation solving. The first eight were one- or two-step equations which the subject solved by undoing (those which had large coefficients) or by substitution on the basis of his number facts. The equations with multiple operations and one occurrence of the unknown were usually solved by totaling the numerical terms on both the left and right sides and then substituting or using the DSOTBS scheme in the resulting one-step equation. Thus, a major change in the solving scheme had occurred when the quasi-equality scheme was altered (by the arithmetic identity approach to equations). Equations with two unknowns were often solved by gathering together the like terms on each side, then using the DSOTBS scheme for the last steps. Not all equations with two occurrences of the unknown seemed to present the same degree of difficulty. Generally, those with two occurrences on the same side seemed easier, for these cases did not involve removing one of the literal terms from a side. In the June posttest, subject did not gather together unlike terms. In the task involving the comparison between equivalent (or non-equivalent) equations, whereas in the pretest he had used all three basic schemes -- quasi-equality, redistribution, and the undoing sub-scheme of the solving

scheme -- in the posttest he tended to evoke primarily the DSOTBS approach to explain the equivalence of two equations.

Long-term effects of teaching sessions on solving scheme. The September posttest asked the same questions as the June posttest. A comparison of the two indicates that one- and two-step equations were still being solved by undoing (for larger coefficients) and by substituting using known number facts (for smaller coefficients). Multistep equations in one unknown were solved by totaling the numbers on the right hand side, totaling groups of numbers on the left hand side, and then successfully using the DSOTBS scheme for the rest. Equations with two occurrences of the unknown proved to be a problem because they tended to evoke the "gathering unlike terms" scheme -- a scheme which, though not used in the June posttest, had been generated and used during the experiment. It seems that once the like terms were gathered, the subject then went on to gather the unlike terms:

9:26:INT: How about this one, $3\underline{a} + 5 + 4\underline{a} = 40$.

9:27:SUB: (Subject wrote: $3\underline{a} + 5 + 4\underline{a} = 40$
$7\underline{a} + 5 = 40$ → 1st error
$12\underline{a} = 40$
$12 - 12\underline{a} = 40 - 12$ → 2nd error
$\underline{a} = 28$) → 3rd error
So, a would equal 28, so it would be 3 times 28 plus 5 plus 4 times 28 equals 40. ... No, I don't think so.

9:28:INT: Why not?

9:29:SUB: Because 28 times 3 is already bigger than 40.

The subject's critic function was activated with this solution. He perceived that this solution was too large for the given equation. The critic seemed to function on the basis of the relative size of the numbers. If the solution attained had been smaller, that is, not grossly off target,

the critic would probably not have been activated (as it had not been activated on other occasions), and the subject would not have bothered to verify the solution. The second error above, placing the subtrahend before the minuend, underlines the non-importance for the subject of the order of two numbers which are being subtracted. This example of arithmetic thinking is another well-known scheme (Davis, 1980), that is, it does not matter which you place first, for you always take the smaller from the larger. Evidence of this scheme was seen on a couple of other occasions during the teaching experiment.

Since the solution was obviously incorrect, the subject re-solved the equation:

$$
\begin{aligned}
& 3\underline{a} + 5 + 4\underline{a} = 40 \\
& 7\underline{a} + 5 = 40 \\
& 7\underline{a} + 5 - 5 = 40 - 5 \\
& 7\underline{a} = 35 \\
\Rightarrow \quad & 7 - 7\underline{a} = 35 - 7 \\
& \underline{a} = 28
\end{aligned}
$$

Misapplication of the rule of doing the same operation to both sides occurred again. This persistent error on the September posttest was also seen in situations where the subject had to eliminate one of the two given occurrences of the unknown from one of the two sides: (9:48)

$$
\begin{aligned}
& 12 + 15\underline{a} - 7 + 6\underline{a} = 4\underline{a} + 107 \\
& 12 - 12 + 15\underline{a} - 7 + 6\underline{a} = 4\underline{a} + 107 - 12 \\
& 15\underline{a} - 7 + 6\underline{a} = 4\underline{a} + 95 \\
& 15\underline{a} - 7 + 7 + 6\underline{a} = 4\underline{a} + 95 + 7 \\
& 21\underline{a} = 4\underline{a} + 102 \\
& 21\underline{a} - 4 = 4\underline{a} - 4 + 102 \\
\Rightarrow \quad & 17\underline{a} = \underline{a} + 102 \\
& 17\underline{a} - 17 = \underline{a} + 102 - 17 \quad \text{(after having considered } 102/17 \text{ and } 102 - 17\text{)} \\
& \underline{a} = 85
\end{aligned}
$$

This sequence led to the ad hoc decision at the end to drop the $\underline{a}$ on the right hand side on order to reach the final goal.

What can we conclude about the status of the solving scheme, three months after the teaching experiment? The difficulties with multistep

equations containing more than one occurrence of the unknown were not resolved -- in view of the performance on the September posttest. The construction and use of the "gathering unlike terms" scheme as a way of overcoming obstacles in the path of producing a solution was an error which seemed to become stronger over the summer months. The rule of "doing the same operation to both sides" facilitated enormously the task of comparing equivalent equations, but as a tool for solving equations its use to the novice seemed quite limited. The uncertainty of knowing exactly what one should do to both sides seemed to end up very often in incorrect choices and lead eventually to the gathering of unlike terms.

Discussion

Davis (1980) has suggested some notions of Information Processing theory which are quite relevant to any discussion of conceptual frameworks. Three of his suggested principles which seem to be applicable to the present discussion are the following: i) discriminations are only as fine as is necessary, ii) accumulation must take place, iii) the highest-level programs must run. To illustrate the first notion, Davis has provided the example of first-grade children learning to add such things as 1 + 2. They have no need to attend to the plus sign; this is the only operation they know. Thus, the input of 1 and 2 leads to the answer 3. The second notion is based on a theory that old ideas are never deleted from memory. They may be used less often, but they continue to exist. The third notion implies that , in the absence of some necessary input, adjustments will be made in order that processing can continue and the final goal can be reached. These three notions help to explain some of the seeming contradictions involved in the coexistence of the main conceptual schemes described in this paper.

Discriminations Are Only as Fine as is Necessary

The use of the redistribution scheme ($\underline{a} + \underline{x} = \underline{b} \Rightarrow \underline{a} + \underline{c} + \underline{x} = \underline{b} - \underline{c}$) would seem to suggest a total lack of attention to the equal sign. Though the origin of the redistribution scheme in algebra novices is unknown, all three of the hypothesized sources exclude any focusing by the child on the notion of equality and the use of the equal sign: (a) the experiences of "sharing" gained outside the school environment (indicating the influence of semantic content on algebraic behavior); (b) overgeneralization of the "adding zero" concept, $\underline{a} + \underline{x} = \underline{b} \Rightarrow \underline{a} + \underline{c} - \underline{c} + \underline{x} = \underline{b}$ (here the focus is on the neutrality of the transformation, rather than on the use of the equal sign); (c) an overgeneralization (erroneous) of the undoing scheme -- since $\underline{a} + \underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{b} - \underline{a}$, then $\underline{c} + \underline{a} + \underline{x} = \underline{b} - \underline{a}$ (here the focus is on the plus on one side and the minus on the other side -- maintaining equivalence is not being attended to). Since it would appear that the plus and minus signs (or rather the notions of adding and subtracting) are the symbols being discriminated in this scheme, it follows that instruction emphasizing the role of the equal sign should help alter the redistribution scheme.

Similarly, the quasi-equality scheme (e.g., $4 + \underline{x} - 2 + 5 = 11 + 3 - 5 \Rightarrow 4 + 4 - 2 + 5 = 11 + 3 - 5 = 9$) can be said to involve a kind of lack of discrimination of the equal sign -- this time, however, it is a lack of discrimination of an equivalence notion of equality. The quasi-equality scheme is based on an operator view of the equal sign wherein it serves as a signal to compute the given operations and write down the answer. Thus, there is no notion of maintaining a balance between right and left sides of the equal sign. The effect of this quasi-equality scheme on the solving scheme is quite clear. In fact, it is only when the quasi-equality scheme

assimilates an equivalence view of the equal sign that the solving of equations with multiple operations on both sides can proceed accurately.

In every one of these three schemes, the equal sign plays a key role. The use of the equal sign in these schemes covers the spectrum from a complete lack of attention (in the redistribution scheme) to an explicit awareness of the need to maintain equivalence between both sides of an equation (the solving scheme). This finding would seem to suggest increasing the focus in the classroom on the equal sign, for it is seen as a key factor underlying the initial learning of algebra.

The issue of discriminations being only as fine as is necessary would seem also to be a factor in the construction of the "gathering unlike terms" scheme. If when novices divide both sides of an equation by the same number (e.g., $7\underline{a} = 42$, $7\underline{a} \div 7 = 42 \div 7$, $\underline{a} = 6$), they are not discriminating the division sign (in $7\underline{a} \div 7$) -- perhaps they do not have to because this is the only sign that has ever appeared in this kind of transformation thus far -- then it is highly possible that when they see $7\underline{a} - 7$ in a later equation, the response is likely to be $\underline{a}$. This inference, based on the validity of $7\underline{a} \div 7 = \underline{a}$, would seem to be one of the most common errors found in algebra. It is an example of what Carry, Lewis, and Bernard (1980) called "deletion operator" errors in their study of algebra errors committed by college students. The possible contribution of the horizontal notation to this error would suggest the use of a vertical notation (e.g., $\frac{7\underline{a}}{7}$), when dividing both sides of an equation by the same number, as an aid in preventing the construction of this overgeneralization.

Accumulation Must Take Place

This information processing notion can account for the coexistence of contradictory schemes in a learner. An example of two contradictory schemes

being used on the same task, one scheme one day (undoing: $\underline{a} + \underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{b} - \underline{a}$) and another scheme a few days later (the Running Total Heuristic of the quasi-equality scheme: $\underline{a} + \underline{x} = \underline{b} \Rightarrow \underline{x} = \underline{b} + \underline{a}$), is seen in the following excerpts from the pretest:

1:398:INT: Let's take this as a model, $\boxed{297 + \underline{a} = 734}$. I'm going to show you four equations and I want you to tell me whether the $\underline{a}$ in those equations will have the same value as the $\underline{a}$ in our model. How about $297 - 297 + \underline{a} = 734 + 297$, will the $\underline{a}$ have the same value?

1:399:SUB: ... (3) ... Nope. No.

1:400:INT: Why not?

1:401:SUB: 'Cause like this is zero plus $\underline{a}$, like these two (the $734 + 297$) are supposed to be like minus here, they're supposed to be $734 - 297$.

The next week:

2:1:INT: If you see this, $\boxed{297 + \underline{a} = 734}$, and I show you another one, $297 - 297 + \underline{a} = 734 + 297$, can you tell just by looking at them if this $\underline{a}$ is going to have the same value as that $\underline{a}$?

2:2:SUB: Um ... (6) ... Well, this is zero, plus $\underline{a}$ equals 734 plus 297. Well, it could be the same thing. It could be, 'cause here's zero, so if you take it off, it will be like plus $\underline{a}$ equals, this is added up after the equal sign, here it's still before.

2:3:INT: So, do you think the $\underline{a}$ will be the same here?

2:4:SUB: Yes.

2:5:INT: as it is there? It will be the same number?

2:6:SUB: Yes.

Similarly, the contradictory redistribution scheme ($\underline{a} + \underline{x} = \underline{b} \Rightarrow \underline{a} + \underline{c} + \underline{x} = \underline{b} - \underline{c}$) and "doing the same operation to both sides" scheme ($\underline{a} + \underline{x} = \underline{b} \Rightarrow \underline{a} + \underline{c} + \underline{x} = \underline{b} + \underline{c}$) were both in evidence on the September posttests.

This information processing notion that old ideas are never deleted from memory is not meant to imply that instruction is powerless in guiding

the learner to construct new schemes and assimilate these to his existing conceptual framework. If an old concept is proved to be false, it is not then forgotten. Presumably it still remains in memory, but now with a tag that it is false. Thus, it should not matter if one calls up the wrong scheme, once the scheme has been sufficiently well falsified.

The approach of constructing arithmetic identities seemed extremely powerful in altering most of the quasi-equality scheme. However, instruction in "doing the same operation to both sides" was not sufficient to prevent the redistribution scheme from gaining dominance in one question of the September posttest. Similarly, the Running Total Heuristic, "It does not matter if you do the operation before the equal sign or after", was invoked in one part of the posttest. Though these two schemes were clearly used less often after instruction than they had been at the beginning of the study, they were still existing in an essentially unaltered state.

The problem would seem to be that both the redistribution scheme and the Running Total Heuristic sub-scheme are alright in certain circumstances (in certain non-symbolic extra-school situations). Thus, they can only be partly falsified -- when related to the interpretation and manipulation of algebraic equations. Similarly, in the Clement (1982) study, the struggle between the label scheme and the numerical variables scheme could not be resolved because the label scheme was clearly appropriate in other situations.

Another facet of this question concerns the influence of semantic content on algebraic behavior. It is generally accepted that students often bring their own daily-life schemes to tasks involving the interpretation and solving of word problems. However, this study indicates that even purely symbolic expressions in mathematics can evoke extra-school schemes. This important finding

has implications for classroom testing, especially for individualized evaluations in which the teacher attempts to uncover a child's understandings of mathematical symbolism. Tasks designed to evaluate understanding, as opposed to those which test skills, seem to call upon a much larger set of schemes, even some schemes which though appropriate outside the school environment are not necessarily going to be helpful or even correct in school math situations. In other words, students' understandings of mathematical expressions--forms which are generally considered to be devoid of semantic content -- may at times reflect thinking which is influenced by the very rich semantic experiences of daily life.

Processing Will Always Continue Until the Goal is Reached

This information processing notion was most clearly seen when subjects were solving multistep equations with several occurrences of the unknown. When faced with unfamiliar situations, caused usually by their insertion on both sides of an inappropriate operation, subjects made several ad-hoc but erroneous decisions in order to reach the final goal of solving the equation, for example:

$$12 + 15\underline{a} - 7 + 6\underline{a} = 4\underline{a} + 107$$
$$12 - 12 + 15\underline{a} - 7 + 6\underline{a} = 4\underline{a} + 107 - 12$$
$$15\underline{a} - 7 + 6\underline{a} = 4\underline{a} + 95$$
$$15\underline{a} - 7 + 7 + 6\underline{a} = 4\underline{a} + 95 + 7$$
$$21\underline{a} = 4\underline{a} + 102$$
$$21\underline{a} - 4 = 4\underline{a} - 4 + 102$$
$$17\underline{a} = \underline{a} + 102$$
$$17\underline{a} - 17 = \underline{a} + 102 - 17$$
$$\underline{a} = 85$$

The above example points out both a strength and a weakness of the rule of "doing the same operation to both sides". This rule is very powerful conceptually. In the study, not only did it facilitate enormously the task of interpreting equivalent equations, it also helped make explicit the idea that each equation in the solving sequence is both balanced in itself and equivalent to the other equations in the chain. However, this

concept seemed less powerful from a procedural point of view. Performing the same operation on both sides is always correct in itself (except multiplication by 0), but it does not lead anywhere if the learner has not chosen a useful transformation to perform. In fact, it often leads to erroneous results for the solver who has not yet discriminated sufficiently well the "gathering of like terms".

The interaction between the rule of doing the same operation to both sides and the erroneous gathering of unlike terms seems to cause difficulty for the algebra novice. The former is self checking on the basis of form; the latter is not. The transformation of $21\underline{a} = 4\underline{a} + 102$ into $21\underline{a} - 4 = 4\underline{a} - 4 + 102$ is correct on the basis of form -- we have subtracted the same quantity from both sides; the transformation of $21\underline{a} - 4 = 4\underline{a} - 4 + 102$ into $17\underline{a} = \underline{a} + 102$ cannot, however, be checked on the basis of form, but rather on the basis of content. In other words, this transformation, $21\underline{a} - 4 \Rightarrow 17\underline{a}$, can only be refuted by substituting a value for $\underline{a}$ (it is important that it be a value other than 1). Lewis (1980) has suggested that students be taught how to do this kind of checking. However, this study indicates that students will not stop to perform this kind of arithmetic check since they do not think that there is anything wrong with the process they are using.

In fact, students rarely check their work. Nevertheless, they do seem to have a critic function. However, it activates only when a solution seems way off the mark. For example, when the subject came up with the solution of 28 to the equation, $3\underline{a} + 5 + 4\underline{a} = 40$, he noticed immediately that 28 had to be incorrect, because "28 times 3 is already bigger than 40". When the domain of equations is whole numbers, the critic (which thinks arithmetically) can use the relative size of the numbers and a few other rules, such as, addition and multiplication make the answers larger and

vice versa for subtraction and division -- rules of thumb which the critic does not have when the domain of algebraic equations is enlarged to include the set of negative integers.

Conclusion

This study uncovered three conceptual schemes existing in 12- and 13-year-olds prior to formal instruction in algebra: the quasi-equality scheme, the redistribution scheme (one which has not been discussed in the research literature up to now), and the solving scheme. A teaching approach based on the gradual expansion of the meaning of the equal sign, by the use of arithmetic identities with multiple operations on both sides, was most effective in altering the main component of the quasi-equality scheme. The teaching experiment was slightly less successful with some aspects of the other schemes: the September posttest indicated the continued (though used less often) presence of the notion, "if we take something from one term, we add it to another" (irrespective of location in the equality)--redistribution; the June posttest, "adding before the equal sign is the same as adding after the equal sign" -- the running total heuristic component of the quasi-equality scheme. Furthermore, the study provided some evidence of the spontaneous generation of the erroneous "gathering unlike terms" scheme (e.g., $7\underline{x} - 7 = \underline{x}$).

The two schemes which seemed resilient to instruction are schemes which are valid outside the symbolic domain of algebra, that is, in extra-school situations, where lack of discrimination of the equal sign is irrelevant. Thus, these schemes continue to exist, with the characteristic that they are appropriate in some situations, inappropriate in others. The task of instruction would then seem to be that of explicitly pointing out to the learner why these schemes are unsuitable within a field where the emphasis is on maintaining an equality and of providing the learner with

effective alternate schemes. The teaching approach adopted in this study seemed quite well suited for achieving the latter goal. Its emphasis on the importance of the equal sign reflects the view that an equivalence notion of equality is a key factor underlying the initial learning of algebra.

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Appendix

Pretest Questions

These are the questions as prepared for the semi-standardized interviews. Minor deviations occurred when individual subjects asked for a clarification, or when the interviewer wished to clarify a response given by the subject.

1. Equation template (each equation written on a separate card): "Can you tell me what number the letter represents and would you also tell me how you figured it out?"

   $6\underline{b} = 24$
   $2\underline{x} - 6 = 4$
   $\underline{y} + 596 = 1282$
   $16\underline{x} - 215 = 265$
   $\underline{n} + 6 = 18$
   $13\underline{x} + 196 = 391$
   $4\underline{c} + 3 = 11$
   $32\underline{a} = 928$
   $37 - \underline{b} = 18$
   $4 + \underline{x} - 2 + 5 = 11 + 3 - 5$
   $30 = \underline{x} + 7$
   $3\underline{a} + 5 + 4\underline{a} = 19$
   $2 \times \underline{c} + 5 = \underline{c} + 8$
   $4\underline{x} + 9 = 7\underline{x}$

2. (a) If $37\underline{x} = 752$, What happens if I add 2 to the value of $37\underline{x}$?
   (b) If $x + 469 = 1351$, what happens if I add 10 to the left side of this?

3. (a) When you see this, $3\underline{a} + 156 = 372$ (ritten on a card), can you think of something to do to that equation that will make $\underline{a}$ change in value?
   (b) That will not make the $\underline{a}$ change?
   (c) How about for $26\underline{c} = 594$?

4. (a) How would you find the missing number in

$$\begin{array}{r} 365 \\ -\ ??? \\ \hline 857 \end{array}$$

   (b) Why?

5. How would you go from equation 1 to equation 2 in the following?
   (a) 1. $3\underline{x} + 375 = 519$
       2. $3\underline{x} = 144$
       } written on one card
   (b) 1. $6\underline{x} = 2\underline{x} + 150$
       2. $6\underline{x} - 2\underline{x} = 2\underline{x} - 2\underline{x} + 150$
       } written on one card

6. Would you please try to group these equations into whatever groups you think are best? And why?
   (On separate cards: $3\underline{a} + 3 = 2\underline{a} + 8$, $3\underline{y} = 17 - 2$, $3\underline{y} + 5 = 6(\underline{x} + 2)$, $6\underline{x} + 4(3\underline{x} - 8) = 4$, $\underline{n} + 652 - 375 = 1348 + 201$, $87 = 5\underline{b} - 3$, $2\underline{b} + 7 = 3\underline{x} - 5$, $\underline{n} + 482 = 1507$)

7. What hits you first when you see this equation?
   (a) $6 + 40\underline{n} - 2 = 32\underline{n} + 12$
   (b) $96 + 35\underline{c} - 58 = 178 \div 2$
   (c) $3 + \underline{x} + 12 - 5 = 27 + 2$
   (d) $2\underline{x} + 3\underline{y} + 12 = 58 + 6$

8. (a) In solving $12\underline{x} - 3 - 2\underline{x} = 23 + 19$, is there something that you must do first? Why or why not?
   (b) How about $2\underline{p} = 63 - 5\underline{p}$?

9. Can you make up an equation which will have the same solution as this one. Show me. Why?
   (a) $37 + \underline{b} = 168$
   (b) $28\underline{y} = 362$
   (c) $179.\underline{a} - 156 = 893$

10. I will show you a card with an equation on it written in red ink (shown in a box here). Then I will show you some other equations (each one on a separate card). Without working out the equations, can you tell me which ones (shown one at a time) have the same solution as the red one? And why? Also, which equations do not have the same solution as the red one and why not. If you cannot tell, just say so.
   (a) $\boxed{560 + \underline{b} = 683}$
   $560 = 683 + \underline{b}$
   $560 + \underline{b} + 125 = 683 + 125$
   $683 = 560 + \underline{b}$
   $683 + 560 = \underline{b}$
   $\underline{b} + 560 + 125 = 683 - 125$
   $\underline{b} + 560 = 683$
   (b) $\boxed{427\underline{n} + 130 = 852}$
   $427 + 130\underline{n} = 852$
   $427 + 130 = 852\underline{n}$
   $852 = 130 + 427\underline{n}$
   $427\underline{n} = 852 + 130$
   (c) $\boxed{297 + \underline{a} = 734}$
   $297 - 297 + \underline{a} = 734 + 297$
   $297 + \underline{a} - 734 = 734 - 734$
   $\underline{a} + 297 - 297 = 734 - 297$
   $297 - 297 + \underline{a} = 734 - 297$
   (d) $\boxed{349 + \underline{p} = 999}$
   $349 + \underline{p} = 999 + \underline{p}$
   $999 = 349 + \underline{p} - \underline{p}$
   $\underline{p} - \underline{p} + 349 = 999 - \underline{p}$
   $349 + \underline{p} - \underline{p} = 999 - \underline{p}$
   (e) $\boxed{\underline{x} - 350 = 502}$
   $350 = 502 - \underline{x}$
   $502 - 350 = \underline{x}$
   $502 = \underline{x} - 350$
   $350 - \underline{x} = 502$
   $\underline{x} + 350 = 502$

11. On the next card is an equation written in red (shown in a box here). Some people started to solve this equation. Did any of them do anything wrong? And why? Their work is in black on the following cards (each equation shown separately):

(a) $\boxed{6 + 39\underline{m} - 4 + 2 = 43}$

$39\underline{m} + 6 - 4 + 2 = 43$

$6 + 39\underline{m} - 2 = 43$

$6 + 35\underline{m} + 2 = 43$

$6 - 6 + 39\underline{m} - 4 + 4 + 2 - 2 = 43 - 6 + 4 - 2$

$39\underline{m} + 4 = 43$

$39\underline{m} + 2 + 2 = 43$

(b) $\boxed{15\underline{x} - 264 = 376}$

$15\underline{x} = 376 + 264$

$\underline{x} - 264 = 376 \div 15$